Control and detection of immersed objects
Part II - Boundary control of a self-propelled body

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Outline

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Maximization of the propulsion efficiency

Motion by self-propulsion

Definition We shall say that a rigid body $\mathcal{S}$ undergoes a *self-propelled motion* in a liquid $\mathcal{L}$ if

1. the total external force acting on $\mathcal{L}$ is identically zero,
2. the total net force and torque, external to the system $\{\mathcal{S}, \mathcal{L}\}$, acting on $\mathcal{S}$, are identically zero.

In the absence of external actions, the forward force that makes the body move - the thrust - is generated by the body, and the motion is due to the interaction of the body's external surface and the fluid in which it is immersed.
In the case of a rigid body that moves by self-propulsion, since the shape of the body is constant during the motion, the thrust is produced because the body generates a nonzero momentum flux through its boundary, or/and moves portions of its boundary.

A spectrum of Reynolds numbers for self-propelled bodies

A large whale swimming at 10 m/s	300 000 000
A tuna swimming at the same speed	30 000 000
A subcompact car at 3 m/s	600 000
A duck flying at 20 m/s	300 000
A model airplane flying at 1 m/s	70 000
A copepod in a speed burst of 0.2 m/s	300
Flapping wings of the smallest flying insects	30
An invertebrate larva, 0.3 mm long, at 1 mm/s	0.3
A human sperm cell at about 0.3 mm/s	0.01

Motion by self-propulsion at low Reynolds number

The problem reduces to: determine v, p, ξ and ω

$$\left.\begin{aligned} \Delta v - \nabla p = 0 \\ \nabla \cdot v = 0 \end{aligned}\right\} \text{ em } \Omega$$

$$v(x) = \xi + \omega \times x + v_*(x) \text{ sobre } \Sigma$$

$$\lim_{|x|\to\infty} v(x) = 0,$$

$$\int_\Sigma T(v,p)n = \int_\Sigma x \times T(v,p)n = 0$$

We assume that v_*, the thrust velocity, is known.

Auxiliary Stokes flows

We will look for a solution (v, p) of the above problem in $\mathrm{span}\{(H^{(i)}, P^{(i)}), i = 1:6\}$, where

$$\left.\begin{aligned} \Delta H^{(i)} - \nabla P^{(i)} = 0 \\ \nabla \cdot H^{(i)} = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$H^{(i)} = \widetilde{e_i} \text{ on } \Sigma$$

$$\lim_{|x| \to \infty} H^{(i)}(x) = 0$$

and

$$\widetilde{e_i} = \begin{cases} e_i, & i = 1, 2, 3 \\ e_{i-3} \times x, & i = 4, 5, 6 \end{cases}$$

Auxiliary Stokes flows (cont.)

Example Let $\Omega = \mathbb{R}^3 \setminus B(0,1)$. Then the pair $(H^{(i)}, P^{(i)})$ defined by

$$H^{(i)}(x) = -\frac{1}{4}\nabla\left(\nabla \cdot \frac{e_i}{|x|}\right) + \frac{3}{4}\left[\frac{e_i}{|x|} + \frac{(e_i \cdot x)x}{|x|^3}\right]$$

and

$$P^{(i)}(x) = \frac{3}{4} e_i \cdot \nabla\left(\frac{1}{|x|}\right)$$

satisfies

$$\left.\begin{aligned} \Delta H^{(i)} - \nabla P^{(i)} &= 0 \\ \nabla \cdot H^{(i)} &= 0 \end{aligned}\right\} \text{ in } \Omega$$

$$H^{(i)} = e_i \text{ on } \Sigma$$

$$\lim_{|x|\to\infty} H^{(i)}(x) = 0$$

The Stokes exterior problem

Theorem Let Σ be of class $C^{1,1}$ and g an element of the *Sobolev space* $H^{3/2}(\Sigma)$. Then the problem

$$\left.\begin{aligned} \Delta v - \nabla p = 0 \\ \nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$v = g \text{ on } \Sigma$$

$$\lim_{|x|\to\infty} v(x) = 0$$

has one and only one solution (v, p) such that $v \in L^s(\Omega)$ $(s \in [6, \infty))$ and $(\nabla v, \nabla p) \in H^1(\Omega)^{3\times 3} \times L^2(\Omega)^3$.

Note that here both the domain of the flow and the velocity on the boundary of this domain are known.

In general, for the computation of the solution, *numerical methods* have to be used.

Decoupling the equations of the liquid and the solid

Due to the linearity of the Stokes equations, it is possible to decouple the motion of the solid from that of the fluid.

Let $(v, p) = (c_i H^{(i)} + \vartheta, c_i P^{(i)} + \pi)$, with $c \in \mathbb{R}^6$ and (ϑ, π) solution of the Stokes problem

$$\left.\begin{aligned} \Delta\vartheta - \nabla\pi &= 0 \\ \nabla\cdot\vartheta &= 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\vartheta = v_* \text{ on } \Sigma$$

$$\lim_{|x|\to\infty} \vartheta(x) = 0$$

Resolution of the self-propulsion problem

We begin by observing that $(v, p) = (c_i H^{(i)} + \vartheta, c_i P^{(i)} + \pi)$ satisfies

$$\left.\begin{aligned} \Delta v - \nabla p = 0 \\ \nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$v(x) = c_i \tilde{e}_i + v_*(x) \text{ on } \Sigma$$

$$\lim_{|x| \to \infty} v(x) = 0$$

for all $c \in \mathbb{R}^6$. Now we see that

$$\int_\Sigma T(v,p)n = \int_\Sigma x \times T(v,p)n = 0 \text{ and } (v,p) = (c_i H^{(i)} + \vartheta, c_i P^{(i)} + \pi)$$

$$\Longleftrightarrow \mathbb{R}c = b$$

with

$$b_i = -\int_\Sigma \tilde{e}_i \cdot T(\vartheta, \pi)n, \quad i = 1:6\,,$$

$$\mathbb{R}_{ij} = \int_\Sigma \tilde{e}_j \cdot T(H^{(i)}, P^{(i)})n, \quad i,j = 1:6.$$

Resistance matrix (or propulsion matrix)

In what follows, we assume that Σ is of class $C^{1,1}$.

The resistance matrix $\mathbb{R}$ is defined by

$$\mathbb{R}_{ij} = \int_\Sigma \tilde{e}_j \cdot T(H^{(i)}, P^{(i)})n, \quad i,j = 1:6,$$

Lemma (i) The resistance matrix can be given by

$$\mathbb{R}_{ij} = \int_\Omega D(H^{(i)}) : D(H^{(j)}), \quad i,j = 1:6\,.$$

(ii) The resistance matrix is symmetric and positive definite.

Remark The matrix $\mathbb{R}$ depends only on the geometry of the solid.

Solution of the self-propulsion problem

Lemma The thrust force and the corresponding torque are given by

$$b_i = -\int_\Sigma e_i \cdot T(\vartheta, \pi)n = -\int_\Sigma v_* \cdot T(H^{(i)}, P^{(i)})n, \quad i = 1:3.$$

and

$$b_i = -\int_\Sigma e_{i-3} \times x \cdot T(\vartheta, \pi)n = -\int_\Sigma v_* \cdot T(H^{(i)}, P^{(i)})n, \quad i = 4:6.$$

Summarizing:

1. Solve the linear system $\mathbb{R}c = b$
2. Take $(v, p) = (c_i H^{(i)} + \vartheta, c_i P^{(i)} + \pi)$, $\xi = c_i e_i$ and $\omega \times x = c_i e_{i-3} \times x$

The thrust space

Question Is a given distribution of velocities v_* in the surface of $\mathcal{S}$ able to propel the body ($V \neq 0$)?
Consider the thrust space

$$\mathcal{T}(\mathcal{S}) := \text{span}\{\mathcal{G}^{(1)}, ..., \mathcal{G}^{(6)}\} \subset L^2(\Sigma)$$

with $\mathcal{G}^{(i)} := T(H^{(i)}, P^{(i)})n$.
The set $\{\mathcal{G}^{(1)}, ..., \mathcal{G}^{(6)}\}$ is linearly independent. In $\mathcal{T}(\mathcal{B})$ we consider the following norm

$$\|\mathcal{G}\| = \|\sum_{i=1}^{6} \alpha_i \mathcal{G}^{(i)}\| := \sum_{i=1}^{6} |\alpha_i|$$

and we denote by $\mathbb{P}$ the projection operator in $\mathcal{T}(\mathcal{S})$.

Relation between the velocity of the body and the thrust velocity

From the previous system for ξ and ω, we conclude that

1. to propel $\mathcal{S}$ with a nonzero velocity we should prescribe boundary velocities such that $\mathbb{P}(v_*) \neq 0$:

$$V \neq 0 \Longleftrightarrow \mathbb{P}(v_*) \neq 0;$$

2. boundary velocities having different projections on $\mathcal{T}(\mathcal{S})$ generate different rigid motions:

$$\mathbb{P}(v_{*1}) \neq \mathbb{P}(v_{*2}) \Longleftrightarrow V_1 \neq V_2;$$

3. the motion of $\mathcal{S}$ is purely translational if and only if the projection of v_* in the subspace $\{\mathcal{G}^{(4)}, \mathcal{G}^{(5)}, \mathcal{G}^{(6)}\}$ is zero;
4. the motion of $\mathcal{S}$ is a pure rotation if and only if the projection of v_* in the subspace $\{\mathcal{G}^{(1)}, \mathcal{G}^{(2)}, \mathcal{G}^{(3)}\}$ is zero.

Equivalent form of the self-propelling conditions

Proposition The conditions

$$\int_\Sigma T(v,p)\cdot n = \int_\Sigma x \times (T(v,p)\cdot n) = 0$$

are equivalent to

$$\int_\Sigma (v_* + V)\cdot \mathcal{G}_i = 0, \quad i = 1:6$$

where $\mathcal{G}^{(i)} := T(H^{(i)}, P^{(i)})n$.

The maximization/minimization problem

- Drag (sometimes called fluid resistance) refers to forces that oppose the relative motion of an object through a fluid.
- Drag forces act in a direction opposite to the oncoming flow velocity. Unlike other resistive forces such as dry friction, which is nearly independent of velocity, drag forces depend on velocity.
- For a solid object moving through a liquid, the drag is the component of hydrodynamic force acting opposite to the direction of the movement. Therefore drag opposes the motion of the object, and in self-propelled body it is overcome by thrust.

The maximization/minimization problem

Our aim is to control v_* in order to propel $\mathcal{S}$ with a prescribed velocity V, with *maximal efficiency*. The efficiency is defined as the ratio

$$\mathcal{E} = \mathcal{W}_{ref}/\mathcal{W}$$

between the work needed to overcome the drag exerted on the body by the fluid $\mathcal{W} = \displaystyle\int_\Sigma (v_* + V) \cdot T(v,p)nds$ and some reference $\mathcal{W}_{ref}$, which can be the work needed for the rigid body to move with constant velocity V with $v_* = 0$. Hence we will consider the problem of *minimizing*

$$\mathcal{W} = \mathcal{W}_{ref}\mathcal{E}^{-1} = \int_\Sigma (v_* + V) \cdot T(v,p)nds = \frac{1}{2}\int_\Omega |D(v)|^2 dx$$

subject to the constraints $\displaystyle\int_\Sigma T(v,p)n = \int_\Sigma x \times (T(v,p)n) = 0.$

Formulation as an Optimization problem

The class $\mathcal{A}$ of admissible functions - the candidates for a minimizer - is, in a first stage, defined by

1. $v \in L^6(\Omega)^3$ and $\nabla v \in L^2(\Omega)^{3\times 3}$,
2. $\nabla \cdot v = 0$ in Ω,
3. $\displaystyle\int_\Sigma v \cdot \mathcal{G}_i = 0,\ i = 1:6.$

Our aim is to find a minimizer for the objective functional

$$I : \mathcal{A} \to \mathbb{R}$$

$$I(v) = \int_\Omega |D(v)|^2 dx,$$

that is, an element $u \in \mathcal{A}$ that satisfies

$$I(u) = \inf_{v \in \mathcal{A}} I(v).$$

Solution of the optimization problem

- $I(v) \geq 0, \forall v \in \mathcal{A}$
- $I(0) = 0$ and $0 \in \mathcal{A}$
- If $u \in \mathcal{A}$ and $I(u) = 0$ then $D(u) = 0 \Leftrightarrow u \in \mathcal{R}$. Since $u \in L^6(\Omega)^3$ then $u = 0$.

Hence this minimization problem has a unique (and trivial!) solution in $\mathcal{A}$, $u = 0$, which corresponds to the boundary velocity $u_* = -V$.

Alternative formulation as an Optimization problem

We may restrict the class of admissible functions, by introducing suitable side conditions which are motivated by physical interest:

- the velocity field is prescribed on a proper subset of the boundary,
- the velocity field at the boundary has a prescribed tangential component,
- the velocity field at the boundary has a prescribed normal component, etc.

In will study the case where the velocity field at the boundary has a prescribed normal component.

Formulation as an Optimization problem

Now the class of admissible functions is composed of those functions satisfying

1. $v \in L^6(\Omega)^3$ and $\nabla v \in L^2(\Omega)^{3\times 3}$,
2. $v \cdot n = \psi$ at Σ,
3. $\nabla \cdot v = 0$ in Ω,
4. $\displaystyle\int_\Sigma v \cdot \mathcal{G}_i = 0,\ i = 1:6,$

and is denoted by $\mathcal{A}_\psi$.

Our aim is to find the minimizer for the functional $I : \mathcal{A}_\psi \to \mathbb{R}$

$$I(v) = \int_\Omega |D(v)|^2 dx = \|D(v)\|^2_{2,\Omega},$$

that is, an element $u \in \mathcal{A}_\psi$ that satisfies

$$I(u) = \inf_{v \in \mathcal{A}_\psi} I(v).$$

Once such a u is obtained, we take the the following velocity field at the boundary $u_* = u|_\Sigma - V$.

Existence and uniqueness of minimizer

Theorem There exists a unique function $u \in \mathcal{A}_\psi$ solving

$$I(u) = \min_{v \in \mathcal{A}_\psi} I(v).$$

Proof 1. Set $m := \inf_{v \in \mathcal{A}_\psi} I(v)$ and select a minimizing sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_\psi$:

$$I(u_k) \longrightarrow m.$$

2. Since $I(u_k)$ is convergent, it is bounded: there exists $M > 0$ such that

$$\|D(u_k)\|_{2,\Omega} \leq M, \forall k \in \mathbb{N}.$$

Consequently, there exists a $\tilde{M} > 0$ such that

$$\|\nabla u_k\|_{2,\Omega}, \ \|u_k\|_{6,\Omega} \leq \tilde{M}, \forall k \in \mathbb{N}.$$

Existence and uniqueness of minimizer

3. There exists a subsequence $\{u_{k'}\}_{k' \in \mathbb{N}}$ and $u \in L^6(\Omega)^3$ with $\nabla u \in L^2(\Omega)^{3\times 3}$ such that

$$D(u_{k'}) \rightharpoonup D(u)$$

$$u_{k'}|_\Sigma \rightarrow u|_\Sigma$$

and $u \in \mathcal{A}_\psi$. Then

$$I(u) \leq \liminf_{k' \rightarrow \infty} I(u_{k'}) = m.$$

Since $u \in \mathcal{A}_\psi$, we have

$$I(u) = m = \min_{v \in \mathcal{A}_\psi} I(v)$$

and existence is proved.

Existence and uniqueness of minimizer

4. It remains to prove uniqueness. Assume u, $\tilde{u} \in \mathcal{A}_\psi$ are both minimizers of I. Then $v := \dfrac{u+\tilde{u}}{2} \in \mathcal{A}_\psi$ and

$$I(v) + \left\| D(\frac{u-\tilde{u}}{2}) \right\|^2_{2,\Omega} = \frac{I(u)+I(\tilde{u})}{2} = m$$

that is

$$I(v) \leq m$$

and since $I(v) < m$ is impossible, we have $J(v) = m$. Therefore $D(\frac{u-\tilde{u}}{2}) = 0$ and this means that

$$u - \tilde{u} \in \mathcal{R}.$$

Since $u - \tilde{u} \in L^6(\Omega)^3$, it must be $u = \tilde{u}$.

First variation, Euler-Lagrange equations

Fix any $v \in \mathcal{A}_0$ and set

$$i(\tau) := I(u + \tau v) = \int_\Omega |D(u + \tau v)|^2 dx \quad (\tau \in \mathbb{R}).$$

Since u is a minimizer of $I(\cdot)$ and $u + \tau v \in \mathcal{A}_\psi$ we observe that $i(\cdot)$ has a minimum at $\tau = 0$. Therefore $i'(0) = 0$. We explicitly compute this derivative, called the first variation by writing

$$\frac{i(\tau) - i(0)}{\tau} = \tau \int_\Omega |D(v)|^2 dx + 2 \int_\Omega D(u) : D(v) dx \quad (\tau \neq 0).$$

Hence

$$i'(0) = 0 \Leftrightarrow \lim_{\tau \to 0} \frac{i(\tau) - i(0)}{\tau} = 0 \Leftrightarrow 2 \int_\Omega D(u) : D(v) dx = 0.$$

First variation, Euler-Lagrange equations

In particular

$$2\int_\Omega D(u):D(v)dx = 0, \quad \forall v \in C_0^\infty(\Omega)^3 \text{ with } \nabla\cdot v = 0.$$

Observe that this is equivalent to

$$\int_\Omega \nabla u : \nabla v dx = 0, \quad \forall v \in C_0^\infty(\Omega)^3 \text{ with } \nabla\cdot v = 0.$$

By classical results for the Stokes equations, there exists $p \in L^2(\Omega)$ such that

$$\Delta u - \nabla p = 0 \text{ in } \Omega$$

which is the Euler-Lagrange equation corresponding to the minimization of the functional $I(v) := \int_\Omega |D(v)|^2 dx$.

First variation, Euler-Lagrange equations

Recall that $u \cdot n = \psi$ on Σ, $\nabla \cdot u = 0$ in Ω and $\displaystyle\int_\Sigma u \cdot \mathcal{G}_i = 0$, $i = 1:6$.

Let $v \in \mathcal{A}_0$. Then

$$\int_\Omega v \cdot (\Delta u - \nabla p) dx = 0$$

which by integration by parts, furnishes

$$\int_\Sigma v \cdot T(u,p) n ds = 0.$$

It can be shown that from $\displaystyle\int_\Sigma u \cdot \mathcal{G}^{(i)} = 0$, $i = 1:6$, and $\displaystyle\int_\Sigma v \cdot T(u,p) n ds = 0$ it follows

$$((T(u,p)n) \times n) \times n = T(u,p)n - n(T(u,p)n) \cdot n \\ \in \mathrm{span}\{(\mathcal{G}^{(i)} \times n) \times n;\ i = 1:6\}.$$

Solution of the optimization problem

1. Solve the problem

$$\left.\begin{aligned} \Delta u - \nabla p = 0 \\ \nabla \cdot u = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$u \cdot n = \psi \text{ on } \Sigma$$

$$((T(u,p)n) \times n) \times n \in \text{span}\{(\mathcal{G}^{(i)} \times n) \times n;\ i = 1:6\}$$

$$\lim_{|x| \to \infty} u(x) = 0$$

$$\int_\Sigma T(v,p)n = \int_\Sigma x \times (T(v,p)n) = 0$$

2. Take $v_* = u|_\Sigma - V$.