Control and detection of immersed objects
Part I - Preparatory results

Ana Leonor Silvestre

Instituto Superior Técnico, Technical University of Lisbon

(Ana.Silvestre@math.ist.utl.pt)

European Summer School in Industrial Mathematics
Università degli Study di Milano, July 18–30, 2011

Outline

Navier-Stokes equations (Claude-Louis Navier - 1822 and George Gabriel Stokes - 1845)

- The Navier-Stokes equations are partial differential equations, where the unknowns are the velocity field and the pressure of a Newtonian fluid.
- The Navier-Stokes equations may be used to model weather, ocean currents, flow around an airfoil and motion of stars inside a galaxy, in the design of aircrafts and cars, the study of blood flow, the design of power stations, the analysis of the effects of pollution, etc. They are used in video games in order to model a wide variety of natural phenomena, including simulations of effects such as water, fire, smoke, etc.
- The problem of existence of regular solutions for the three-dimensional equations, properly formulated, is part of the problems selected by the *Clay Mathematics Institute*, which awards the prize of one million dollars to its resolution.

Navier-Stokes equations for an incompressible fluid - The classical problem

Given a bounded, fixed (for the moment) domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, $T > 0$, a density ρ, a viscosity μ, an external force F, v_* and v_0, find the velocity $v = v(t,x) = v_i(t,x)e_i$ and the pressure $p = p(t,x)$ of the fluid, defined in $[0,T] \times \Omega$, satisfying

$$\left.\begin{array}{l} \rho(\partial_t v + (v \cdot \nabla_x)v) = \mu \Delta_x v - \nabla_x p + \rho F \\ \nabla_x \cdot v = 0 \end{array}\right\} \text{ in } (0,T) \times \Omega$$

$$v(t,x) = v_*(t,x) \text{ on } (0,T) \times \partial\Omega \text{ (boundary condition)}$$

$$v(0,x) = v_0(x),\ x \in \Omega \text{ (initial condition)}$$

Remark $((v \cdot \nabla_x)v)_i = v_j \dfrac{\partial v_i}{\partial x_j} \quad (i = 1 : n)$

Einstein notation When an index variable appears twice in a single term it means that we are summing over all of its possible values. Typically, in applications, the values of the indices are 1,2 e 3, representing the 3 dimensions in the Euclidean space.

Navier-Stokes equations for an incompressible fluid

Remarks $v = v(t,x)$, $\quad v : [0,T] \times \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$

$$\Delta_x v_i = \sum_{j=1}^{n} \frac{\partial^2 v_i}{\partial x_j^2} \quad (i = 1:n)$$

$$((v \cdot \nabla_x)v)_i = v_j \frac{\partial v_i}{\partial x_j} = \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j} \quad (i = 1:n)$$

$$\rho(\partial_t v + (v \cdot \nabla_x)v) = \mu \Delta_x v - \nabla_x p + \rho F$$

$$\iff \rho(\partial_t v_i + v_j \frac{\partial v_i}{\partial x_j}) = \mu \Delta_x v_i - \frac{\partial p}{\partial x_i} + \rho F_i \quad (i = 1:n)$$

$$\nabla_x \cdot v = 0 \iff \frac{\partial v_j}{\partial x_j} = 0 \iff \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} = 0$$

$$v_i(t,x) = (v_*)_i(t,x) \text{ on } (0,T) \times \partial\Omega \quad (i = 1:n)$$

$$v_i(0,x) = (v_0)_i(x),\ x \in \Omega \quad (i = 1:n)$$

Example: Plane Couette Flow

Consider a horizontal flow between two parallel fixed planes located at $x_3 = \pm d$. The motion is driven by a constant horizontal pressure gradient

$$\frac{\partial p}{\partial x_1} = -C$$

where C is a given constant. It is reasonable to look for a steady motion where the velocity field v is directed parallel to the planes and depends only on the vertical coordinate x_3: $v(x) = V(x_3)e_1$.
The function V, which describes the way in which the velocity varies within the layer, is calculated by imposing that $v(x) = V(x_3)e_1$ solves the Navier-Stokes equations with $\partial v/\partial t \equiv 0$ (steady flow) and $F = -ge_3$, where g is the magnitude of the acceleration of gravity.
On the boundary of the domain, we consider no-slip boundary conditions, expressed by $V(-d) = V(d) = 0$.

Plane Couette Flow

Since $v(x) = V(x_3)e_1$, we have

$$\nabla \cdot v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial V}{\partial x_1} = 0$$

$$(v \cdot \nabla)v = V \frac{\partial V}{\partial x_1} e_1 = 0$$

so that the Navier-Stokes equations reduce to $\mu \Delta v - \nabla p = -\rho F$. This is equivalent to the following three scalar equations

$$\mu \frac{d^2 V}{dx_3^2} + C = 0$$

$$-\frac{\partial p}{\partial x_2} = 0$$

$$-\frac{\partial p}{\partial x_3} = \rho g$$

Plane Couette Flow

From the equations

$$-\frac{\partial p}{\partial x_1} = C, \quad -\frac{\partial p}{\partial x_2} = 0, \quad -\frac{\partial p}{\partial x_3} = \rho g$$

we obtain

$$p(x) = -\rho g x_3 - C x_1 + p_0 \quad (p_0 \text{ constant}).$$

Likewise, from

$$\mu \frac{d^2 V}{dx_3^2} + C = 0$$

we obtain

$$V(x_3) = -\frac{C}{2\mu} x_3^2 + A x_3 + B$$

where the constants A and B are obtained by imposing the no-slip boundary conditions $V(-d) = V(d) = 0$. We thus find $A = 0$ and $B = \frac{Cd^2}{2\mu}$, so that $V(x_3) = \frac{C}{2\mu}(d^2 - x_3^2)$.

Navier-Stokes equations - Related problems

- The most relevant case in applications is $\Omega \subset \mathbb{R}^n$ with $n = 3$.
- One can consider more specific situations: bi-dimensional flows ($n = 2$); stationary motions, creeping flow, periodic motions (in time or space); flows in unbounded domains (for instance, exterior domains, tubes)...
- One can consider different definitions of solutions according to the class of functions to which the solutions should belong: classical solutions, *strong solutions*, weak solutions, very weak solutions, statistical solutions, ...
- The functional framework includes: Hölder spaces, Sobolev spaces, Besov spaces, Lorentz spaces, ...

Nondimensionalization of the Navier-Stokes equations. Change of variables

We introduce the characteristic parameters τ, l, ϑ, ς in order to define the dimensionless variables

$$t' = \frac{t}{\tau}, \quad x' = \frac{x}{l}, \quad f'(t', x') = \frac{f(\tau\, t', l\, x')}{\varsigma}$$

and

$$v'(t', x') = \frac{v(\tau\, t', l\, x')}{\vartheta}$$

$$p'(t', x') = \frac{\rho\, l\, p(\tau\, t', l\, x')}{\mu\, \vartheta}.$$

For the pressure, we get

$$(\nabla_{x'} p')(t', x') = \frac{\rho\, l^2}{\mu\, \vartheta} (\nabla_x p)(\tau\, t', l\, x')$$

Change of variables in the Navier-Stokes equations

For the velocity:

$$(\partial_{t'} v')(t', x') = \frac{\tau}{\vartheta}(\partial_t v)(\tau\, t', l\, x')$$

$$(\nabla_{x'} v')(t', x') = \frac{l}{\vartheta}(\nabla_x v)(\tau\, t', l\, x')$$

$$(\nabla_{x'} \cdot v')(t', x') = \frac{l}{\vartheta}(\nabla_x \cdot v)(\tau\, t', l\, x')$$

$$(\Delta_{x'} v')(t', x') = \frac{l^2}{\vartheta}(\Delta_x v)(\tau\, t', l\, x')$$

$$((v' \cdot \nabla_{x'}) v')(t', x') = \frac{l}{\vartheta^2}((v \cdot \nabla_x) v)(\tau\, t', l\, x')$$

Nondimensionalization of the Navier-Stokes Equations. Reynolds number and Strouhal number

Then (v', p') satisfies

$$\left.\begin{array}{l} \frac{\rho\vartheta}{\tau}\partial_{t'}v' + \frac{\rho\vartheta^2}{l}(v'\cdot\nabla)v' = \frac{\mu\,\vartheta}{l^2}(\Delta v' - \nabla p') + \rho\varsigma f' \\ \nabla\cdot v' = 0 \end{array}\right\} \text{ in } (0, T')\times\Omega'$$

where $T' = \frac{T}{\tau}$, $\Omega' = \{x' = \frac{x}{l};\ x\in\Omega\}$.

The Reynolds number is $\mathrm{Re} = \dfrac{\rho\, l\, \vartheta}{\mu}$ and the Strouhal number is $\mathrm{S} = \dfrac{l}{\tau\,\vartheta}$.

Using these dimensionless numbers, we can write

$$\left.\begin{array}{l} \mathrm{Re}\,\mathrm{S}\partial_{t'}v' + \mathrm{Re}(v'\cdot\nabla)v' = \Delta v' - \nabla p' + \frac{l\,\varsigma\,\mathrm{Re}}{\vartheta^2}f' \\ \nabla\cdot v' = 0 \end{array}\right\} \text{ in } (0, T')\times\Omega'$$

Examples of Reynolds numbers

A large whale swimming at 10 m/s	300 000 000
A tuna swimming at the same speed	30 000 000
A subcompact car at 3 m/s	600 000
A duck flying at 20 m/s	300 000
A model airplane flying at 1 m/s	70 000
A copepod in a speed burst of 0.2 m/s	300
Flapping wings of the smallest flying insects	30
An invertebrate larva, 0.3 mm long, at 1 mm/s	0.3
A human sperm cell at about 0.3 mm/s	0.01

Stationary Navier-Stokes equations

When $S \approx 0$ (for instance, τ is very high) in

$$\left.\begin{array}{l} \mathrm{Re}\,\mathrm{S}\,\partial_t v + \mathrm{Re}(v\cdot\nabla)v = \Delta v - \nabla p + \mathrm{Re}F \\ \nabla\cdot v = 0 \end{array}\right\} \text{ in } (0,T)\times\Omega$$

we obtain the stationary Navier-Stokes equations

$$\left.\begin{array}{l} \Delta v - \nabla p = \mathrm{Re}\,(v\cdot\nabla)v - \mathrm{Re}\,F \\ \nabla\cdot v = 0 \end{array}\right\} \text{ in } \Omega$$

which should be complemented with boundary conditions, for instance

$$v = v_* \text{ on } \partial\Omega$$

if Ω is bounded. Since $\nabla\cdot v = 0$ in Ω, by the Divergence Theorem, we conclude that v_* must satisfy

$$\int_{\partial\Omega} v_*\cdot n dS = 0.$$

Stokes Equations

When $Re \approx 0$ (for instance, the viscosity μ is very high) in

$$\left.\begin{array}{l} \mathrm{Re}\,\mathrm{S}\,\partial_t v + \mathrm{Re}(v\cdot\nabla)v = \Delta v - \nabla p + \mathrm{Re}F \\ \nabla\cdot v = 0 \end{array}\right\} \text{ in } (0,T)\times\Omega$$

we can use the Stokes approximation

$$\left.\begin{array}{l} \Delta v - \nabla p = 0 \\ \nabla\cdot v = 0 \end{array}\right\} \text{ in } \Omega$$

which describes creeping flow. This system must be complemented with a boundary condition, for instance

$$v = v_* \text{ on } \partial\Omega$$

if Ω is bounded.

Outline

Function spaces appropriate for the theory of partial differential equations

- Sobolev spaces
- Trace of a function

Lebesgue spaces

Assume $\Omega \subseteq \mathbb{R}^n$ is an open set and $1 \leq p \leq \infty$. If $f : \Omega \to \mathbb{R}$ is measurable, we define

$$\|f\|_{p,\Omega} = \begin{cases} \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \text{ess sup}_{x \in \Omega} |f(x)|, & p = \infty \end{cases}$$

We define $L^p(\Omega)$ to be the linear space of all measurable functions $f : U \to \mathbb{R}$ for which $\|f\|_{p,\Omega} < \infty$. Then $L^p(\Omega)$ is a Banach space, provided we identify two functions which agree a.e.

Sobolev spaces

The Sobolev spaces are function spaces whose members have *weak derivatives* of various orders lying in various L^p spaces.
Suppose $u, v \in L^1_{loc}(\Omega)$ and α is a multiindex. We say that v is the α^{th}-weak derivative of u, written

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} v \varphi dx$$

for all *test functions* $\varphi \in C_0^\infty(\Omega)$. In other words, if we are given u and if there happens to exist a function v which satisfies the above identity for all φ, we say that $D^\alpha u = v$ in the weak sense. If there does not exist such a function v, then u does not possess a weak α^{th}-partial derivative.

Sobolev spaces

Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. The Sobolev space $W^{k,p}(\Omega)$ consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.
If $p = 2$, we usually write

$$H^k(\Omega) = W^{k,2}(\Omega)$$

If $u \in W^{k,p}(\Omega)$, we define its norm to be

$$\|u\|_{k,p,\Omega} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u|, & p = \infty \end{cases}$$

We denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.
Since differentiation may be performed in Sobolev spaces, this is an appropriate setting for the theory of partial differential equations.

Traces

In order to deal with non-homogeneous boundary values associated with a partial differential equation, we have to give a meaning to the expression u restricted to $\partial\Omega$. This is clear if $u \in C(\overline{\Omega})$. Since $\partial\Omega$ is a set of measure zero, functions in $L^p(\Omega)$ cannot in general have well-defined boundary values. For a function $u \in W^{1,p}(\Omega)$ the notion of *trace operator* resolves this problem.

Theorem

Take $1 \leq p < \infty$. Let $\partial\Omega$ be the graph of a Lipschitz continuous function. There exists a uniquely determined continuous linear mapping $Tr : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that

$$Tr(u) = u|_{\partial\Omega} \quad \textit{for all } u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega).$$

The space $W^{1-\frac{1}{p},p}(\partial\Omega)$ is identified with $Tr(W^{1,p}(\Omega))$. For instance, $H^{\frac{1}{2}}(\partial\Omega) = Tr(H^1(\Omega))$.

Outline

Introduction to fluid-structure interaction

- Formulation of the fluid-rigid body interaction problem in an inertial reference frame (the domain of the fluid is an unknown of the problem)
- Formulation in a reference frame attached to the body (the domain of the fluid becomes fixed and known)
- Stokes Approximation (the problem is linear)

Equations of motion in an inertial reference frame

Flow of a viscous liquid around a rigid body

$$\left.\begin{array}{l}\rho(\partial_t u + (u\cdot\nabla)u) = \mu\Delta u - \nabla p + \rho F_{\mathcal{L}} \\ \nabla\cdot u = 0\end{array}\right\} \text{ in } \cup_{t\in(0,T)}\{t\}\times\Omega(t),$$

$$u(t,y) = u_*(t,y) + U(t,y) \text{ on } \cup_{t\in(0,T)}\{t\}\times\partial\Omega(t),$$

$$\lim_{|y|\to\infty} u(t,y) = 0,\ t\in(0,T)$$

$$u(0,y) = u_0(y),\ y\in\Omega$$

Here

$$U(t,y) = \eta(t) + \varpi(t)\times(y - y_C(t))$$

is the velocity of the rigid body, with

η a velocity of the center of mass : $y_C'(t) = \eta(t)$

ϖ a angular velocity

Description of the domain $\Omega(t)$ in terms of η and ϖ

In what follows, $\Omega := \Omega(0)$.

Recall that the solid moves with velocity

$$U(t,y) = \eta(t) + \varpi(t) \times (y - y_C(t))$$

This velocity field generates a transformation $\chi(U)$ such that the domain occupied by the fluid is

$$\Omega(t) = \chi(U)(t,\Omega) = \{y \in \mathbb{R}^3 : y = \chi(U)(t,x),\ x \in \Omega\}$$

and

$$\begin{cases} \dfrac{d\chi(U)}{dt}(t,x) = U(t,\chi(U)(t,x)) \\ \chi(U)(0,x) = x \end{cases}$$

Without loss of generality, we will assume $y_C(0) = 0$.

The rotation (or orientation) matrix $R(t)$. Description of $\Omega(t)$

Proposition We have $\chi(U)(t,x) = R(t)x + y_C(t)$, with $R(t)$ given by

$$\begin{cases} \dfrac{dR}{dt} = W(\varpi)R \\ R(0) = \mathbb{I} \end{cases}, \qquad W(\varpi) = \begin{bmatrix} 0 & -\varpi_3 & \varpi_2 \\ \varpi_3 & 0 & -\varpi_1 \\ -\varpi_2 & \varpi_1 & 0 \end{bmatrix}.$$

Hence, the domain occupied by the fluid is given by

$$\Omega(t) = \{y \in \mathbb{R}^3 : \; y = R(t)x + y_C(t), \; x \in \Omega\}$$

$SO(3) := \{A \in \mathbb{R}^{3\times 3};\; A^T A = AA^T = \mathbb{I},\, \det A = 1\}$

Proposition $R(t) \in SO(3)$, $\forall t \in [0, T]$.

Remark: $\dfrac{dR}{dt} R^\top a = \varpi \times a, \quad \forall a \in \mathbb{R}^3$

Equations of motion in an inertial frame (cont.)

Motion of a rigid body in a viscous liquid

- Newton laws

$$m\frac{d\eta}{dt} = F_{\mathcal{S}} - \int_{\Sigma(t)} [\mathcal{T}(u,q)N - \rho(u_* + U)u_* \cdot N,$$
$$\frac{d(\mathcal{I}\varpi)}{dt} = M_{\mathcal{S}} - \int_{\Sigma(t)} (y - y_C) \times [\mathcal{T}(u,q)N - \rho(u_* + U)u_* \cdot N],\ t \in (0,T)$$
$$\eta(0) = \eta_0,\ \varpi(0) = \varpi_0$$

In these equations

$$\mathcal{T}(u,q) := \mu(\nabla u + (\nabla u)^T) - q\mathbb{I} = 2\mu D(u) - q\mathbb{I}$$
$$\Sigma(t) := \partial\Omega(t) = \partial\Omega^c(t) \text{ com } \Omega^c(t) := \mathbb{R}^3 \setminus \Omega(t),$$
$$m = \int_{\Omega^c} \varrho(y)dy = \int_{\Omega^c(t)} \varrho(R^\top(t)(y - y_C(t)))dy$$ is the mass of the solid

$$\mathcal{I}_{ij}(t) = \int_{\Omega^c(t)} \varrho(R^\top(t)(y - y_C(t)))[\delta_{ij}|y - y_C(t)|^2 - (y - y_C(t))_i (y - y_C(t))_j]dy$$ is the inertia matrix

Formulation of fluid-structure interaction problem

Given ρ, μ, $F_{\mathcal{L}}$, u_*, $F_{\mathcal{S}}$, $M_{\mathcal{S}}$ and the initial conditions, find u, q, η and ϖ such that

$$\left.\begin{array}{l}\rho(\partial_t u + (u\cdot\nabla)u) = \mu\Delta u - \nabla q + \rho F_{\mathcal{L}} \\ \nabla\cdot u = 0\end{array}\right\} \text{ in } \cup_{t\in(0,T)}\{t\}\times\Omega(t),$$

$$u(t,y) = u_*(t,y) + U(t,y) \text{ on } \cup_{t\in(0,T)}\{t\}\times\partial\Omega(t),$$

$$\lim_{|y|\to\infty} u(t,y) = 0,\ t\in(0,T)$$

$$m\frac{d\eta}{dt} = F_{\mathcal{S}} - \int_{\Sigma(t)} [\mathcal{T}(u,q)N - \rho(u_* + U)u_*\cdot N],$$

$$\frac{d(\mathcal{I}\varpi)}{dt} = M_{\mathcal{S}} - \int_{\Sigma(t)} (y-y_C)\times[\mathcal{T}(u,q)N - \rho(u_*+U)u_*\cdot N],\ t\in(0,T)$$

$$u(0,y) = u_0(y),\ y\in\Omega;\ \eta(0) = \eta_0,\ \varpi(0) = \varpi_0$$

where $\Omega(t) = \{y\in\mathbb{R}^3 :\ y = R(t)x + y_C(t),\ x\in\Omega\}$

Change of variables in the equations of motion

Recall that

$$\Omega(t) = \chi(U)(t,\Omega) = \{y \in \mathbb{R}^3 : y = y_C(t) + R(t)x,\ x \in \Omega\}$$

Since, for each $t \in [0, T]$, $\chi(U)_t := \chi(U)(t,\cdot)$ is a difeomorphism between Ω and $\Omega(t)$, we consider the change of variables

1. $x = \chi(U)_t^{-1}(y) = R^\top(t)(y - y_C(t))$

   Advantage: $\chi(U)_t^{-1}(\Omega(t)) = \Omega$, $\forall t \in [0, T]$ and the region occupied by the fluid becomes: $[0, T] \times \Omega$

2. $$\begin{aligned} v(x,t) &= J_{\chi(U)_t^{-1}}(\chi(U)(t,x))u(t,\chi(U)(t,x)) \\ &= R^\top(t)u(t, y_C(t) + R(t)x) \end{aligned}$$

3. $p(t,x) = q(t,\chi(U)(t,x)) = q(t, y_C(t) + R(t)x)$

4. $\xi(t) = R(t)^\top \eta(t)$, $\omega(t) = R(t)^\top \varpi(t)$

   Observe that $y_C(t) = \displaystyle\int_0^t R(s)\xi(s)ds.$

Change of variables in the equations of motion

Recall that

$$v(t,x) = R^\top(t)u\left(t, R(t)x + \int_0^t R(s)\xi(s)ds\right),$$

$$p(t,x) = q\left(t, R(t)x + \int_0^t R(s)\xi(s)ds\right),$$

$$V(t,x) = \xi(t) + \omega(t) \times x \; com \; \xi(t) = R^\top(t)\eta(t) \; e \; \omega(t) = R^\top(t)\varpi(t).$$

and that

$$\frac{dR^\top}{dt}Ra = a \times \omega, \quad \forall a \in \mathbb{R}^3.$$

We have

$$\nabla p(t,x) = R^\top(t)(\nabla q)\left(t, R(t)x + \int_0^t R(s)\xi(s)ds\right)$$

Change of variables in the equations of motion

And for the velocity of the liquid:

$$\frac{\partial v}{\partial t}(t,x) = R^\top(t)\frac{\partial u}{\partial t}\left(t, R(t)x + \int_0^t R(s)\cdot\xi(s)ds\right)$$
$$+(V(t,x)\cdot\nabla)v(t,x) - \omega(t)\times v(t,x)$$

$$\Delta v(t,x) = R^\top(t)\Delta u\left(t, R(t)x + \int_0^t R(s)\cdot\xi(s)ds\right)$$

$$\nabla\cdot v(t,x) = \nabla\cdot u\left(t, R(t)x + \int_0^t R(s)\cdot\xi(s)ds\right)$$

$$(v\cdot\nabla)v(t,x) = R^\top(t)(u\cdot\nabla)u\left(t, R(t)x + \int_0^t R(s)\cdot\xi(s)ds\right)$$

Change of variables in the equations of motion

Observe that the normal at Σ is given by

$$n(x) = R^{\top}(t)N\left(t, R(t)x + \int_0^t R(s)\cdot\xi(s)ds\right)$$

and that the inertia matrix is transformed into a matrix I

$$I_{ij} = (R(t)^{\top}\mathcal{I}(t)R(t))_{ij} = \int_{\Omega^c} \varrho(y)(\delta_{ij}|y|^2 - y_i y_j)dy$$

which is independent of time, symmetric and positive definite.

Equations of motion in a reference frame attached to $\mathcal{S}$

For the liquid

$$\left.\begin{array}{l}\rho\partial_t v = \mu\Delta v - \nabla p + \rho[((V - v)\cdot\nabla)v - \omega \times v + R^\top F_{\mathcal{L}}] \\ \nabla\cdot v = 0\end{array}\right\} \text{in} \Omega \times (0, T)$$

$$v = V + v_* \text{ on } \Sigma \times (0, T)$$

$$\lim_{|x|\to\infty} v(x,t) = 0, \ \ t \in (0, T)$$

$$v(0, x) = u_0(x), \ y \in \Omega$$

For the rigid body

$$m\frac{d\xi}{dt} + m\omega \times \xi = R^\top F_{\mathcal{S}} - \int_\Sigma [T(v,p)n - \rho(v_* + V)v_* \cdot n],$$

$$I\frac{d\omega}{dt} + \omega \times (I\omega) = R^\top M_{\mathcal{S}} - \int_\Sigma x \times [T(v,p)n - \rho(v_* + V)v_* \cdot n], t \in (0, T)$$

$$\frac{dR^\top}{dt} = -W(\omega)R^\top$$

$$\xi(0) = \xi_0, \quad \omega(0) = \omega_0, \quad R(0) = \mathbb{I}$$

Equations of motion of the system rigid body-liquid

Find v, p, ξ, ω and R such that

$$\left.\begin{aligned} &\rho\partial_t v = \mu\Delta v - \nabla p + \rho[((V-v)\cdot\nabla)v - \omega\times v + R^\top F_{\mathcal{L}}] \\ &\nabla\cdot v = 0 \end{aligned}\right\} \text{ in } \Omega\times(0,T)$$

$$v = v_* + V \text{ on } \Sigma\times(0,T)$$

$$\lim_{|x|\to\infty} v(x,t) = 0, \;\; t\in(0,T)$$

$$m\frac{d\xi}{dt} + m\omega\times\xi = R^\top F_{\mathcal{S}} - \int_\Sigma [T(v,p)n - \rho(v_* + V)v_*\cdot n],$$

$$I\frac{d\omega}{dt} + \omega\times(I\omega) = R^\top M_{\mathcal{S}} - \int_\Sigma x\times[T(v,p)n - \rho(v_* + V)v_*\cdot n], t\in(0,T)$$

$$\frac{dR^\top}{dt} = -W(\omega)R^\top$$

$$v(0,x) = v_0(x), \; y\in\Omega$$

$$\xi(0) = \xi_0, \quad \omega(0) = \omega_0, \quad R(0) = \mathbb{I}$$

Motion by self-propulsion

Definition We shall say that a rigid body $\mathcal{S}$ undergoes a *self-propelled motion* in a liquid $\mathcal{L}$ if

1. the total external force acting on $\mathcal{L}$ is identically zero,
2. the total net force and torque, external to $\{\mathcal{S}, \mathcal{L}\}$, acting on $\mathcal{S}$ are identically zero.

As examples of bodies that move by self-propulsion, we can refer planes, rockets and submarines, and, in the microscopic realm, minute organisms like ciliates and flagellates. The self-propelled motion may occur because of a nonzero momentum flux generated by the body on its boundary, as in a rocket, or because the body moves tangentially portions of its boundary. This latter case typically occurs in micro-organisms.

Motion by self-propulsion

In the absence of external actions, the forward force that makes the body move - the thrust - is generated by the body, and the motion is due to the interaction of the body's external surface and the fluid in which it is immersed.

In the case of a rigid body that moves by self-propulsion, since the shape of the body is constant during the motion, the thrust is produced because the body generates a nonzero momentum flux through its boundary, or/and moves portions of its boundary.

Hence, in order to describe a motion by self-propulsion, we assume in the preceding equations that $F_{\mathcal{L}} = 0$, $F_{\mathcal{S}} = M_{\mathcal{S}} = 0$ and $v_* \neq 0$.

Motion by self-propulsion

The problem is: find v, p, ξ and ω such that

$$\left.\begin{array}{l} \rho\partial_t v = \mu\Delta v - \nabla p + \rho[((V - v)\cdot\nabla)v - \omega\times v] \\ \nabla\cdot v = 0 \end{array}\right\} \text{ in } \Omega\times(0,T)$$

$$v = v_* + V \text{ on } \Sigma\times(0,T)$$

$$\lim_{|x|\to\infty} v(x,t) = 0,\ \ t\in(0,T)$$

$$m\frac{d\xi}{dt} + m\omega\times\xi = -\int_\Sigma [T(v,p)n - \rho(v_* + V)v_*\cdot n],$$

$$I\frac{d\omega}{dt} + \omega\times(I\omega) = -\int_\Sigma x\times[T(v,p)n - \rho(v_* + V)v_*\cdot n],\ t\in(0,T)$$

$$v(0,x) = v_0(x),\ y\in\Omega$$

$$\xi(0) = \xi_0, \quad \omega(0) = \omega_0$$

Nondimensionalization of the equations

If we also adimensionalize the mass and the inertia matrix of the solid, we get

$$\mathrm{Re}\,\mathrm{S}\, m\frac{d\xi}{dt} + \mathrm{Re}\, m\omega \times \xi = -\int_\Sigma [T(v,p)n - \mathrm{Re}(v_* + V)v_* \cdot n],$$

$$\mathrm{Re}\,\mathrm{S}\, I\frac{d\omega}{dt} + \mathrm{Re}\,\omega \times (I\cdot\omega)$$

$$= -\int_\Sigma x \times [T(v,p)n - \mathrm{Re}(v_* + V)v_* \cdot n],\ t \in (0,T),$$

$$\xi(0) = \xi_0, \quad \omega(0) = \omega_0$$

Self propulsion at low Reynolds number

When $\mathrm{Re} \approx 0$, we obtain

$$\left.\begin{aligned} \Delta v - \nabla p = 0 \\ \nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$v = v_* + V \text{ at } \Sigma$$

$$\lim_{|x|\to\infty} v(x) = 0,$$

$$\int_\Sigma T(v,p) \cdot n = 0$$

$$\int_\Sigma x \times T(v,p) \cdot n = 0$$

The problems is to find v, p, V, with $V(x) = \xi + \omega \times x$, knowing v_*.