Control and detection of immersed objects
Part IV - An inverse obstacle problem

Ana Leonor Silvestre, IST, TULisbon

(Ana.Silvestre@math.ist.utl.pt)

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Outline

The direct problem

Stokes equations describing the slow steady flow of an incompressible viscous fluid in the absence of external forces

$$\left\{ \begin{array}{l} \left.\begin{array}{l} \Delta u - \nabla p = 0 \\ \nabla \cdot u = 0 \end{array}\right\} \text{ in } \mathcal{F} \\ u = u_*, \text{ on } \partial\mathcal{F} \end{array}\right. \iff \left\{ \begin{array}{l} \left.\begin{array}{l} \nabla \cdot T(u,p) = 0 \\ \nabla \cdot u = 0 \end{array}\right\} \text{ in } \mathcal{F} \\ u = u_*, \text{ on } \partial\mathcal{F} \end{array}\right.$$

Given

- the bounded domain $\mathcal{F}$ occupied by the fluid
- the velocity u_* on the boundary, satisfying the non-flux condition $\int_{\partial\mathcal{F}} u_* \cdot n \, ds = 0$ (consequence of $\nabla \cdot u = 0$ in $\mathcal{F}$),

find

- the velocity field $u = (u_1, u_2)$ of the fluid
- the pressure p of the fluid,

satisfying the Stokes system.

Linear, well-posed problem

The inverse problem

An inaccessible rigid body, represented by the set $\overline{D}$, is immersed in a Stokes fluid that fills a domain $\Omega \setminus \overline{D}$.
We assume that there are no external forces acting on the fluid and the solid is at rest.

Our aim is to determine D via a stress measurement on $\Gamma := \partial\Omega$, resulting from an imposed distribution of velocities u_* on Γ.
Recall that the Cauchy stress tensor associated to the Stokes flow (u, p) in $\Omega \setminus \overline{D}$ is

$$T(u, p) := \nabla u + \nabla u^\top - pI,$$

where I is the identity tensor. The corresponding stress vector acting on Γ is the vector

$$t_* := T(u, p)n_{|\Gamma}.$$

Mathematical formulation

Given

- the velocity u_* on the outer boundary Γ, satisfying
  $$\int_\Gamma u_* \cdot n \, ds = 0,$$
- the stress vector t_* measured on Γ,

find

- the (unknown!!) connected bounded domain D

such that

$$\begin{cases} \left.\begin{array}{l} \Delta u - \nabla p = 0 \\ \nabla \cdot u = 0 \end{array}\right\} \text{in } \Omega \setminus \overline{D} \\ u = 0 \text{ on } \partial D \\ u = u_* \text{ on } \Gamma \\ T(u,p) \cdot n = t_* \text{ on } \Gamma \end{cases}$$

Nonlinear, ill-posed problem

Main questions

The main steps in the study of the inverse problem are:

- **Identifiability**. Does the observation on the accessible boundary uniquely determines the immersed body?
- **Stability**. Do small perturbations of the measurements give rise to small perturbations of the shape and location of the body?
- **Reconstruction**. How to determine numerically the immersed body? The objective is to build an efficient method to find a good approximation of the unknown domain.

Identifiability

We will look for the unknown domain in the set $\mathcal{G}_{ad}$ of admissible geometries, formed by simply connected bounded domains D with a locally Lipschitz boundary.

Let

$$H^s_\sigma(\Gamma) := \left\{ v \in (H^s(\Gamma))^2 : \int_\Gamma v \cdot n \, ds = 0 \right\}$$

Theorem

Let Γ be locally Lipschitz and $u_* \in H^{1/2}_\sigma(\Gamma) \setminus \{0\}$. Assume that $D^{(1)}, D^{(2)} \in \mathcal{G}_{ad}$ and let $(u^{(1)}, p^{(1)})$ and $(u^{(2)}, p^{(2)})$ be the solutions of the Stokes problems in $\Omega \setminus \overline{D^{(1)}}$ and $\Omega \setminus \overline{D^{(2)}}$, respectively, both with the boundary condition u_* on Γ. If $t^{(1)}_* = t^{(2)}_*$ on Γ then $D^{(1)} = D^{(2)}$.

Formulation of the inverse problem as an optimization problem

As a consequence of the previous theorem, the inverse problem can be formulated as a minimization problem.

Theorem

Let Γ be C^2 and $u_* \in H^{3/2}_\sigma(\Gamma) \setminus \{0\}$. The problem of finding

$$arg\ min_{\mathcal{D}\in\mathcal{G}_{ad}} \|T(u,p)n - T(\tilde{u},\tilde{p})n\|^2_{(L^2(\Gamma))^2},$$

with (u,p) and $(\tilde{u},\tilde{p})$ satisfying

$$\begin{cases} \left.\begin{array}{l} \Delta u - \nabla p = 0 \\ \nabla\cdot u = 0 \end{array}\right\} in\ \Omega\setminus\overline{D} \\ u = 0\ \ at\ \partial D \\ u = u_*\ \ at\ \Gamma \end{cases} \quad and \quad \begin{cases} \left.\begin{array}{l} \Delta \tilde{u} - \nabla \tilde{p} = 0 \\ \nabla\cdot \tilde{u} = 0 \end{array}\right\} in\ \Omega\setminus\overline{\mathcal{D}} \\ \tilde{u} = 0\ \ at\ \partial \mathcal{D} \\ \tilde{u} = u_*\ \ at\ \Gamma \end{cases}$$

has a unique solution.

Approximating boundary curves

The problem of finding the unknown domain D is equivalent to finding $\gamma := \partial D$.
Let

$$\mathcal{B}_k := \begin{cases} \{1\}, \ k = 0 \\ \{1, \sin(t), \cos(t), ..., \sin(kt), \cos(kt)\}, \ k > 0 \end{cases}$$

We will look for approximating curves $\widetilde{\gamma} = \widetilde{\partial D}$ parameterized by

$$\widetilde{\gamma}(t) = r(t)(\cos(t)e_1 + \sin(t)e_2) + c_1 e_1 + c_2 e_2 \quad (t \in [0, 2\pi])$$

with $r \in \mathrm{span}\ \mathcal{B}_k$, for some $k \in \mathbb{N}_0$. Thus, r is defined by $\alpha \in \mathbb{R}^{2k+1}$, such that

$$r(t) = \sum_{i=0}^{k} \alpha_{2i} \cos(it) + \sum_{i=1}^{k} \alpha_{2i-1} \sin(it).$$

The measured data

In order to find the $2k+3$ coefficients $c \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}^{2k+1}$ that define $\widetilde{\gamma}$, we fix a sufficiently large number of observation points x_j, $j = 1 : J$, on the accessible boundary of the fluid domain.

The stress vector $t_* := T(u,p)n$ will be measured in the points $x_j \in \Gamma$, $j = 1 : J$, and may be affected by random noise:

$$\widetilde{t_*(x_j)} = (1+\varepsilon_j)t_*(x_j) \quad (j = 1 : J)\,.$$

Note that these are $2J$ scalar measurements.

The discrete objective functions

For $k \in \mathbb{N}_0$ such that $2k+3 < 2J$, we consider a discrete version of the objective function of the previous theorem

$$F : \mathbb{R}^2 \times \mathbb{R}^{2k+1} \to \mathbb{R}_0^+$$

$$F(c^{(k)}, \alpha^{(k)}) = \sum_{j=1}^{J} |\widetilde{t_*(x_j)} - t_*^{(k)}(x_j)|^2$$

where $t_*^{(k)}(x_j) = T(u^{(k)}, p^{(k)})(x_j) n^{(k)}(x_j)$ and $(u^{(k)}, p^{(k)})$ satisfies

$$\begin{cases} \left.\begin{array}{l} \Delta u^{(k)} - \nabla p^{(k)} = 0 \\ \nabla \cdot u^{(k)} = 0 \end{array}\right\} \text{in } \Omega \setminus \overline{D^{(k)}} \\ u^{(k)} = 0, \text{ at } \gamma^{(k)} \text{ and } u^{(k)} = u_*, \text{ at } \Gamma \end{cases}$$

and $\gamma^{(k)} = \partial D^{(k)}$ is defined by $c^{(k)} \in \mathbb{R}^2$ and

$$r^{(k)}(t) = \sum_{i=0}^{k} \alpha_{2i}^{(k)} \cos(it) + \sum_{i=1}^{k} \alpha_{2i-1}^{(k)} \sin(it).$$

The nonlinear least squares problems

In order to find $c^{(k)}$ and $\alpha^{(k)}$ (for k fixed!!), we have to solve the nonlinear problem of finding

$$\arg\min_{(c,\alpha)\in\mathbb{R}^2\times\mathbb{R}^{2k+1}} F(c,\alpha)$$

with $F : \mathbb{R}^2 \times \mathbb{R}^{2k+1} \longrightarrow \mathbb{R}_0^+$

$$F(c^{(k)},\alpha^{(k)}) = \sum_{j=1}^{J} |\widetilde{t_*(x_j)} - t_*^{(k)}(x_j)|^2$$

The residual vector $R \in \mathbb{R}^{2J}$ associated to F is

$$R_j = \begin{cases} e_1 \cdot (\widetilde{t_*(x_j)} - t_*^{(k)}(x_j)), \ j = 1 : J, \\ e_2 \cdot (\widetilde{t_*(x_j)} - t_*^{(k)}(x_j)), \ j = J+1 : 2J. \end{cases}$$

This problem will be solved by the Levenberg-Marquardt Algorithm.

Application of the Levenberg-Marquardt Algorithm

In each iteration step of the LMA, a current approximation $(c_{prev}, \alpha_{prev})$ is replaced by a new estimate $(c_{new}, \alpha_{new}) = (c_{prev} + \delta_c, \alpha_{prev} + \delta_\alpha)$ where $\delta := (\delta_c, \delta_\alpha)$ is obtained by solving the linear system

$$[A(c_{prev}, \alpha_{prev}) + \lambda I]\delta = J_R(c_{prev}, \alpha_{prev})^T R(c_{prev}, \alpha_{prev})$$

where

- $A(c_{prev}, \alpha_{prev}) := J_R(c_{prev}, \alpha_{prev})^T J_R(c_{prev}, \alpha_{prev})$
- $J_R = J_R(c, \alpha)$ is the Jacobian matrix of the residual vector $R = R(c, \alpha)$ w.r.t. c and α
- λ is the regularization parameter

The main difficulty is the computation of the Jacobian matrix $J_R(c, \alpha)$.

Computation of the Jacobian matrix $J_R(c,\alpha)$

The Jacobian matrix at a point (c,α), $J_R(c,\alpha)$, is approximated using a centered finite-difference scheme.

Example: for $j = 1 : J$

$$\begin{aligned}\frac{\partial R_j}{\partial \alpha_l}(c,\alpha) &\approx \frac{R_j(c,\alpha + he_l) - R_j(c,\alpha - he_l)}{2h}\\ &= \frac{e_1 \cdot t_*(c,\alpha + he_l)(x_j) - e_1 \cdot t_*(c,\alpha - he_l)(x_j)}{2h}\end{aligned}$$

where

$$t_*(c,\alpha + he_l) = T(u(c,\alpha + he_l), p(c,\alpha + he_l))n(c,\alpha + he_l)$$

is calculated from $(u(c,\alpha + he_l), p(c,\alpha + he_l))$ that solves the (direct) Stokes problem in the domain defined by the outer boundary Γ and the inner boundary $\gamma = \gamma(c,\alpha + he_l)$. The outward unit normal to $\gamma(c,\alpha + he_l)$ is $n(c,\alpha + he_l)$.

Computation of the Jacobian matrix $J_R(c,\alpha)$

Recall that

$$\gamma(c,\alpha+he_l)(t) = r(c,\alpha+he_l)(t)(\cos(t)e_1+\sin(t)e_2)+c_1e_1+c_2e_2.$$

If l is odd, then

$$r(c,\alpha+he_l)(t) = \sum_{i=0}^{k}\alpha_{2i}\cos(it)+\sum_{i=1,2i-1\neq l}^{k}\alpha_{2i-1}\sin(it)+(\alpha_l+h)\sin(\frac{l+1}{2}t).$$

Then $(u(c,\alpha+he_l), p(c,\alpha+he_l))$ is the solution of

$$\begin{cases} \left.\begin{array}{l}\Delta u - \nabla p = 0\\ \nabla\cdot u = 0\end{array}\right\} \text{ in } \Omega\setminus\overline{D(c,\alpha+he_l)} \\ u = 0 \text{ at } \gamma(c,\alpha+he_l) \text{ and } u = u_* \text{ at } \Gamma\end{cases}$$

These direct problems can be solved by the Method of Fundamental Solutions.

Reconstruction of the immersed body: the iterative process

The value of k and the corresponding curve $\gamma^{(k)}$, are updated in several steps.

1. Start with $k = 0$ in order to obtain a first approximation for the size and location of γ. The LMA for finding $c^{(0)}, \alpha^{(0)}$ is initialized with a value $(\widetilde{c^{(0)}}, \widetilde{\alpha^{(0)}})$ that guarantees that the corresponding circle $\widetilde{\gamma^{(0)}}$ is contained in Ω;
2. Consider $0 < k_1 < k_2 < \ldots < k_S$ such that $2k_S + 3 < 2J$. Once a certain $(c^{(k_i)}, \alpha^{(k_i)})$ is obtained, the update $(c^{(k_{i+1})}, \alpha^{(k_{i+1})})$ is searched by the LMA described for fixed k, initializing the method with $(c^{(k_i)}, \alpha^{(k_i)})$.

The stopping criterion used for each k is $\dfrac{|R^{(i+1)} - R^{(i)}|}{|R^{(i)}|} < \varepsilon$. Here $R^{(i)} \in \mathbb{R}^{2J}$ is the residual vector in the iteration i in $\mathcal{B}_k$. The value of ε is prescribed and can be related with the level of noise in the measurements.

Example 1

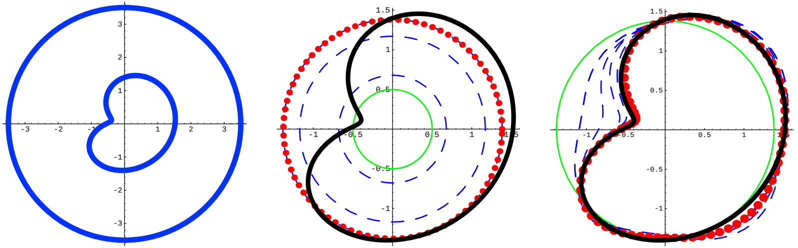
Parametrization of (the unknown) γ :

$$\gamma(t) = \frac{1.4 + 1.3\cos(t) + 0.1\sin(2t)}{1 + 0.8\cos(t)}(\cos(t)e_1 + \sin(t)e_2)$$

Velocity imposed on Γ : $u_*(x_1, x_2) = \frac{1}{12}(x_2 e_1 + x_1 e_2)$

Measurements of the stress in 50 observation points, with a level of noise of 10%

Starting from $\partial B(0, 0.5)$, after 4 iterations in $\mathcal{B}_0$ (center) and 5 iterations in $\mathcal{B}_4$ (right), we obtained



Example 2

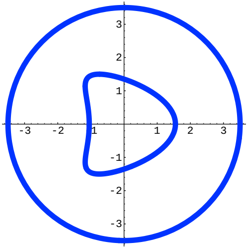
Parametrization of (the unknown) γ:

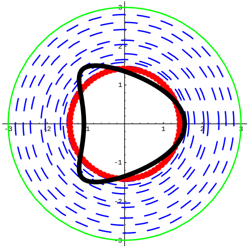
$\gamma(t) = (1.3\cos(t) + 0.5\cos(2t) - 0.25)e_1 + 1.5\sin(t)e_2$

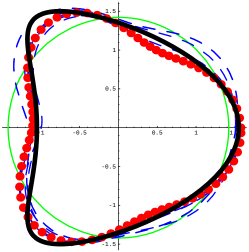
Velocity imposed on Γ : $u_*(x_1, x_2) = \frac{1}{12}(x_2 e_1 + x_1 e_2)$

Measurements of the stress in 40 observation points, with a level of noise of 10%

The results obtained in $\mathcal{B}_0$ (12 iterations) and in $\mathcal{B}_4$ (7 iterations), starting from $\partial B(0, 3.0)$ are following







Example 3

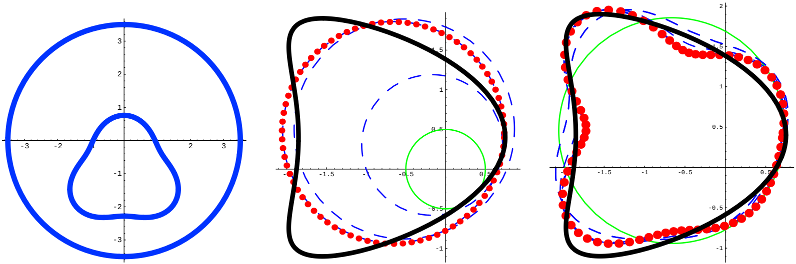
Parametrization of γ :

$\gamma(t) = (1.3\cos(t) + 0.5\cos(2t) - 1.05)e_1 + (0.4 + 1.5\sin(t))e_2$

Velocity imposed on Γ : $u_*(x_1, x_2) = e_2$

Measurements of the stress in 40 observation points, with a level of noise of 10%

The results obtained in $\mathcal{B}_0$ (4 iterations) and in $\mathcal{B}_4$ (6 iterations), starting from $\partial B(0, 0.5)$ are



Example 4

Parametrization of γ :

$\gamma(t) = \gamma(t) = (1.5 - 0.2\sin(3t))(\cos(t)e_1 + \sin(t)e_2) - 1.0e_2$

Velocity imposed on Γ : $u_*(x_1, x_2) = e_2$

Measurements of the stress in 40 observation points, with a level of noise of 10%

The results obtained in $\mathcal{B}_0$ (8 iterations) and in $\mathcal{B}_4$ (5 iterations), starting from $\partial B(0, 0.5)$ are